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USE OF THE PARAMETRIC REPRESENTATION METHOD IN REVEALING THE ROOT STRUCTURE AND HOPF BIFURCATION

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Abstract

In practical applications of dynamical systems, it is often necessary to determine the number and the stability of the stationary states. The parametric respresentation method is a useful tool in such problems. Consider the two parameter families of functions: $f(x) = u_0 + u_1 x + g(x)$, where u_0 and u_1 are the parameters. We are interested in the number of zeros as well as in the stability. We want to determine the "stable region" on the parameter plane, where the real parts of the roots of f are negative. The D-curve (along which the discriminant of f is zero) helps us. We applied the method to the cases of cubic and quartic equation, giving pictorial meaning to the root structure. In this respect, the R-curves and the I-curves (along which the sum or difference, respectively, of two zeros is constant) also have a significance. Using these concepts, we established a relation between the $(n-1)$th Routh–Hurwitz condition and the Hopf bifurcation.

1. Introduction

The system of autonomous differential equations

$$\dot{x} = F(u;x) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \tag{1}$$

is used in wide areas of natural sciences. Here, the system has n degrees of freedom, x_i are the state variables ($i = 1, \ldots, n$), and u_j are the control parameters ($j = 1, \ldots, m$).

Changing the values of the control parameters u_j, the qualitative character of the solutions often remains unchanged, and changes only at certain exceptional values of u_j. The qualitative change of solutions is called *bifurcation,* and the parameter values at which the bifurcations take place are the bifurcation points. If we have two control parameters ($m = 2$), the set of bifurcation points constitutes a curve in the generic case: we will use the term *bifurcation diagram.*

The bifurcation diagram divides the parameter plane (u_1, u_2) into separate regions; the qualitative behaviour of the solutions is the same within any region.

One of the most important bifurcation problems is the determination of the number of equilibria (stationary points). This number changes only at the *singularity set* defined by the system

$$F(u;x) = 0 \quad \frac{\partial F}{\partial x}(u;x) = 0. \tag{2}$$

In practice, the determination of the bifurcation diagram is not an easy task. For example, the elimination of the state variables x_i from eq. (2), in principle, leads to a relation between the control parameters, but this elimination is often very tiresome, and even if the elimination was feasible, the shape of the bifurcation diagram could not be seen in this way in the vast majority of cases.

The method of parametric representation provides us with the equation of the bifurcation diagram in a parametric form. Here, one of the state variables serves as a parameter, and the equation of the bifurcation diagram takes the form

$$u_1 = u_1(x), \qquad u_2 = u_2(x). \tag{3}$$

This method was formerly used for special chemical problems ([8,12,13]; the last work lists further references). The term "parametric representation method" was introduced by Gilmore [9]. Also, we refer to other relevant works which contain systematic approaches and general results [1–3,5,10,11].

Local investigation of the system (1) requires study of the solutions of "algebraic" (i.e. *not* differential) equations in dependence on the control parameters. These equations may arise in at least two different ways:

(a) Put zero for the left-hand side of (1) and eliminate the state variables except one; in this way, we obtain an equation which determines the location of the equilibria.

(b) Calculate the characteristic polynomial at a certain equilibrium; in this way, we obtain an equation which determines the character of the equilibrium in question.

In this paper, we investigate the solutions of the equation

$$f(x) \equiv u_0 + u_1 x + g(x) = 0, \tag{4}$$

where u_0 and u_1 are the control parameters. Farkas et al. [6] investigated this problem in detail, but that work focused on equations determining equilibria. Here, we pay special attention to type (b) (i.e. when (4) is a characteristic equation), and consequently, we will not only be interested in real, but also in complex solutions. Furthermore, we are interested in the problems of stability: we want to determine the "stable" region in the plane (u_0, u_1) where

$$\mathrm{Re}(x) < 0 \tag{5}$$

is valid for *all* solutions of (4).

2. The discriminant curve (D-curve)

DEFINITION

To eq. (4) we assign its discriminant curve (*D-curve* for short) given by the following parametric equations:

$$u_0 = u_0(x) \equiv x g'(x) - g(x),$$
$$u_1 = u_1(x) \equiv -g'(x) \,. \tag{6}$$

For the points of the D-curve, eq. (2) has a multiple solution, and just this multiple solution x is the parameter along the D-curve. Substituting (6) into (4), we see that

$$f(x) = 0, \quad f'(x) = 0. \tag{7}$$

In other words, the D-curve defined by (6) is the singularity set of the function f [10]. For the case where g is a polynomial, the D-curve is the locus of the parameter-pairs (u_0, u_1) for which the discriminant D of eq. (4) is zero (see the appendix).

From the D-curve, we can easily gain all the information about the real roots of (4). Indeed, disregarding some exceptional cases, the following statements were proved [6]:

(i) The number of real roots changes only if we cross the D-curve and, therefore, this curve divides the parameter plane into regions with different numbers of real roots.

(ii) The straight line tangential to the D-curve at the point assigned to the value x is the locus of the parameters (u_0, u_1) for which x is a solution of (4). Therefore, all the real solutions belonging to any given parameter-pair (u_0, u_1) can be obtained by drawing tangential straight lines to the D-curve from the given point.

(iii) The slope of the D-curve at the point $(u_0(x), u_1(x))$ is:

$$\mathrm{d}u_0/\mathrm{d}u_1 = -x. \tag{8}$$

(iv) The curvature of the D-curve can be given by the relation

$$\mathrm{d}^2u_0/\mathrm{d}u_1^2 = 1/g''(x). \tag{9}$$

The D-curve may have some peculiar points, cusps and self-crossing points.

Cusps

These are determined by the equations:

$$f(x) = 0, \quad f'(x) = 0, \quad f''(x) = 0. \tag{10}$$

This means that the cusp points are the points of the D-curve for which

$$g''(x) = 0. \tag{11}$$

At a cusp, the D-curve reverses its direction and changes its curvature. In the inner part of the cusp, the number of real roots is greater than outside.

Self-crossing points

It may occur that the same point belongs to two different values of x by (6). In this case, the D-curve intersects itself. To the self-crossing point (u_0, u_1), the equation $f(x) = 0$ has at least two multiple solutions.

It is yet an open problem how to give the self-crossing points in a simple way for the general case.

The self-crossing points have special significance with respect to complex roots. Namely, for some complex value x, the formulas in (6) may produce real u_0 and u_1. The points (u_0, u_1) given in this way (that is, generated by complex x's) constitute a set which we will call *the complex supplement of the D-curve*. The self-crossing points as well as the points of the complex supplement are discrete points in the generic case. In the case of polynomial $f(x)$, the complex solutions occur in conjugate pairs. Hence, the self-crossing points and the points of the complex supplement can be defined by the same definition, namely, the requirement for the existence of two multiple roots. In section 6, we will give explicit formulas and show that the complex supplement is a continuation of the self-crossing point for quartic equations.

3. The R-curves and I-curves

DEFINITION

To eq. (4), we define the family of R-curves and I-curves by the formulas in parametric form:

$$\begin{aligned} u_0 &= -g(R+I) + [g(R+I) - g(R-I)](R+I)/2I, \\ u_1 &= -[g(R+I) - g(R-I)]/2I. \end{aligned} \tag{12}$$

For an R-curve, R is constant and I is the parameter of the curve, while for an I-curve, I is constant and R is the parameter of the curve.

Remarks

(i) Notice that the D-curve can be considered as a special I-curve, namely the one assigned to the value $I = 0$. (Strictly speaking, for $I = 0$ the r.h.s. of (12) is not defined, but we can consider the limit case $I \rightarrow 0$.)

(ii) The point (u_0, u_1) given by (12) is the unique point for which $R+I$ and $R-I$ are the solutions of the equation $f(x)=0$ provided that $I \neq 0$.

(iii) If I is purely imaginary ($I = \mathrm{i}y$, $y \in \mathbb{R}$), then the point (u_0, u_1) is the unique point to which the complex number $R + \mathrm{i}y$ (and also $R - \mathrm{i}y$, of course) is a solution. (Compare: the locus of the points in the parameter plane for which a certain real value x is a solution is a straight line.)

(iv) An R-curve is the locus of the points for which the sum of two solutions is constant (this sum if R).

(v) An I-curve is the locus of the points for which the difference of two solutions is constant (this difference is I).

(vi) For complex values R and I, the relations in (12) produce real u_0 and real u_1 only in some exceptional cases. Besides the trivial case, when R and I are both real, there is another typical case for real u_0, u_1: the case of a conjugate pair of complex solutions, i.e. when $I = \mathrm{i}y$ ($y \in \mathbb{R}$).

(vii) Hopf bifurcation can occur only if the characteristic polynomial has a purely imaginary pair of roots. Therefore, the Hopf bifurcation diagram is a special R-curve: $R = 0$ and $I = \mathrm{i}y$ ($y \in \mathbb{R}$). Next, we discuss this case in detail.

4. The case $R = 0$: Hopf bifurcation

This special case can be treated in a simple way. Let us consider the points of the plane (u_0, u_1) for which the sum of two solutions is zero or, in other words, together with a solution I, its negative $-I$ is also a solution:

$$\begin{aligned} f(I) &= u_0 + u_1 I + g(I) = 0, \\ f(-I) &= u_0 - u_1 I + g(-I) = 0. \end{aligned} \tag{13}$$

By addition and subtraction of these equations, we obtain the desired curve in parametric form:

$$\begin{aligned} u_0 &= -[g(I) + g(-I)]/2I, \\ u_1 &= -[g(I) - g(-I)]/2I. \end{aligned} \tag{14}$$

Note that on the r.h.s. there are even functions of I. We can consider (14) as a parametric expression for a certain curve H^*, and we can use I^2 as the parameter of the curve given by (14). This curve consists of two parts:

(a) Real part: $I^2 > 0$. Along this part, the sum of two real solutions is zero.

(b) Imaginary part: $I^2 < 0$. Along this part, the sum of two purely imaginary solutions is zero. We will call this part the *H-curve* or Hopf curve, because Hopf bifurcation can occur only if we cross this part of the curve.

These two parts of the curve organically join at the point belonging to $I = 0$.

These results can be used in stability analysis. A system may lose its stability in two typical ways:

(1) A single real solution changes its sign. In our case, this may occur along the line $u_0 = 0$.
(2) The real part of a complex conjugate pair of solutions changes its sign. This bifurcation (Hopf bifurcation) takes place along the H-curve.

In stability investigations, the Routh–Hurwitz criterion is widely used. The following theorem establishes a relation between the Routh–Hurwitz criterion and the above results.

THEOREM

The system of eqs. (13) has a solution I if and only if

$$\Delta_{n-1} = 0, \tag{15}$$

where Δ_{n-1} is the $(n-1)$th principal minor of the Routh–Hurwitz matrix. (See the proof and the details in the appendix.)

This theorem means that the Hopf bifurcation problem is essentially the same as the problem of the $(n-1)$th Routh–Hurwitz condition.

5. Cubic equations

The parametric representation method gives us a simple pictorial description for the solutions of the cubic equation

$$f(x) \equiv u_0 + u_1 x + u_2 x^2 + x^3 = 0. \tag{16}$$

Consider the solutions as functions of u_0 and u_1; u_2 is assumed to be a positive constant.

From (6), the parametric equations of the D-curve are:

$$\begin{aligned} u_0 &= u_0(x) = x^2(2x + u_2), \\ u_1 &= u_1(x) = -x(3x + 2u_2). \end{aligned} \tag{17}$$

Using these formulas with the general results given in section 2, one can readily see the qualitative shape of the D-curve (fig. 1). Both $u_0(x)$ and $u_1(x)$ have a maximum at $x = -u_2/3$, where the D-curve has a cusp.

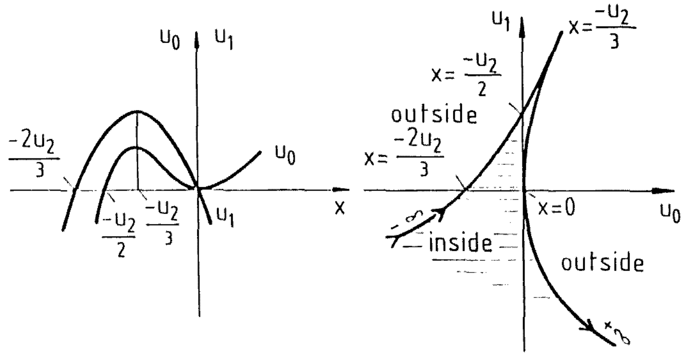

Fig. 1.

The variable x in (17) is the parameter of the curve. We characterize the points of the D-curve by the values of this parameter x. The arrow indicates the direction of increasing x. *When we speak about "point x_1", we mean the point* $(u_0(x), u_1(x))$*!*.

The D-curve divides the plane (u_0, u_1) into two parts: an "inside" part and an "outside" part (related to the cusp).

The straight line tangential to the D-curve at x_1 intersects the D-curve at the point $x = -(u_2 + x_1)/2$. To see this, we recall that at the intersection point x, the cubic equation has a double solution x and also has a solution x_1, and consequently $x + x + x_1 = -u_2$.

To determine the solutions belonging to a certain point of the parameter plane, we can use straight lines tangential to the D-curve. There are two typical situations:

(I) The point $P = (u_0, u_1)$ is the "inside" (fig. 2). Then we can draw three tangential lines, and we obtain three real solutions at the tangent points x_1, x_2, x_3. It is interesting to note that the intersection point (y_i) with the D-curve is just the mean value of the two other corresponding solutions.

(II) The point $P = (u_0, u_1)$ is "outside" (fig. 2). In this case, we can draw only one tangential to the D-curve. The tangent point x_1 is just the unique real solution. The tangential straight line intersects the D-curve at a certain value $x = R$. This value is just the real part of the other two solutions x_2 and x_3. In order to give a pictorial meaning to the imaginary part of the complex solutions, we remark that the tangential straight line in question is the R-curve belonging to the value R above. The parametric equation of the R-curves is given in (12). The second equation of (12) takes the form

$$u_1 = -(2u_2R + 3R^2 + I^2) \tag{18}$$

in the case of cubic eq. (16). Therefore, this equation gives u_1 as a function of I along the tangential straight line. At the intersection point (of the D-curve and the tangential line, fig. 2), $I = 0$; therefore,

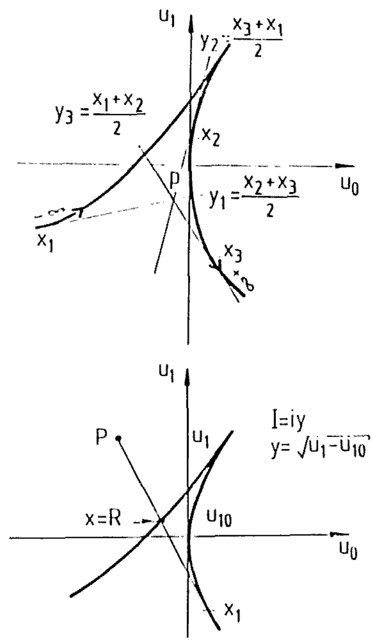

Fig. 2. Graphic determination of the solutions of a cubic equation. Case (I): three real solutions, case (II): one real solution.

$$u_{10} = -(2u_2R + 3R^2),$$

where u_{10} is the u_1-coordinate of the intersection point. Hence,

$$I = \pm\sqrt{u_1 - u_{10}}\,. \tag{19}$$

Now let us turn to the problem of stability. The H*-curve is given from (14):

$$u_0 = -u_2I^2, \quad u_1 = -I^2. \tag{20}$$

This is a straight line going through the origin and tangent to the D-curve at the point $-u_2$ (fig. 3). I^2 can be considered the parameter of the H*-curve. $I^2 = 0$ at the origin and $I^2 < 0$ in the first quadrant. The semi-infinite straight line

$$\{(u_0, u_1)\,|\,u_0 = -u_2I^2;\ u_1 = -I^2;\ I^2 < 0\}$$

is the Hopf curve. Above this curve the real part of the complex solutions is negative. According to fig. 3, we can recognize the following regions with respect to the sign of the real part of the solutions:

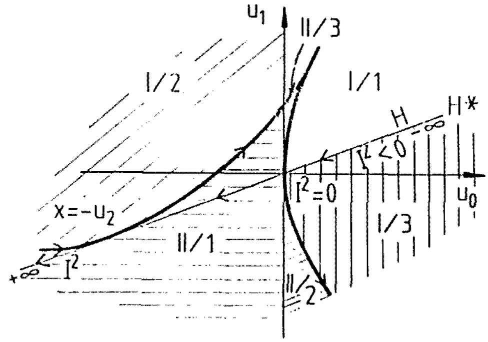

Fig. 3.

I. Outside region

Region	Sign of x_1	Sign of $\mathrm{Re}\,x_2$
I/1. above H-curve, right	$-$	$-$
I/2. above H-curve, left	$+$	$-$
I/3. below H-curve, right	$-$	$+$

II. Inside region

Region	Sign of x_1	Sign of x_2	Sign of x_3
II/1. $u_0 < 0$	$-$	$-$	$+$
II/2. $u_0 > 0$ and $u_1 < 0$	$-$	$+$	$+$
II/3. $u_0 > 0$ and $u_1 > 0$	$-$	$-$	$-$

In this way, we obtain the stable region of the cubic equation. It is in the first quadrant bordered by the axis u_1 and the H-curve $u_0 = u_1 u_2$ that is the union of the regions I/1 and II/3 (fig. 4).

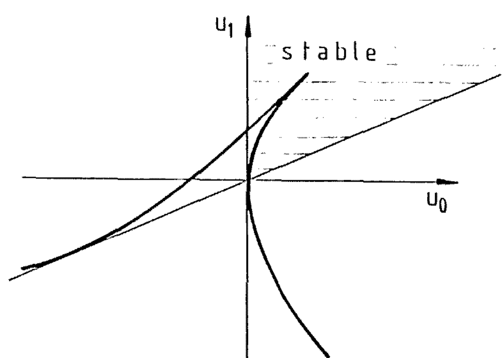

Fig. 4.

6. Quartic equations

The quartic equation in general form is:

$$a_0 + a_1 y + a_2 y^2 + a_3 y^3 + y^4 = 0.$$

With the transformation $y = a_3 x$, the equation takes the form:

$$u_0 + u_1 x + u_2 x^2 + x^3 + x^4 = 0.$$

This transformation does not mean a loss of generality (provided that $a_3 \neq 0$), and has a practical advantage compared to the more often used transformation which cancels the cubic term, namely, for this transformation the relations between the original parameters a_i and the new ones u_i are very simple. We define the polynomials g and f by

$$\begin{aligned} g(x) &= u_2 x^2 + x^3 + x^4, \\ f(x) &= u_0 + u_1 x + u_2 x^2 + x^3 + x^4. \end{aligned} \tag{21}$$

First, we shall study the D-curve. We can obtain the parametric form of the D-curve from (6):

$$u_0 = x^2(u_2 + 2x + 3x^2), \quad u_1 = -x(2u_2 + 3x + 4x^2).$$

The cusp points are determined by (11), $g''(x) = 0$, in our case:

$$u_2 + 3c + 6c^2 = 0,$$

where c is the parameter of the cusp point. Thus, $c_{1,2} = (-3 \pm \sqrt{3}d)/12$, where $d = \sqrt{(3 - 8u_2)}$. (We can see that in the case of the quartic equation, the D-curve has two cusp points or it has no cusp point.)

Now, let us study the self-crossing point, the singularity point of the third order (the case of a fourfold root) and the complex supplementary point. The common feature of these points is that two double roots exist for them (the case of a fourfold root can be considered as a special degenerate case). For this reason, we shall use the term TDR-point (Two Double Root) for all of them. The TDR-points are characterized by the formula:

$$f(x) = (x - x_1)^2 (x - x_2)^2. \tag{22}$$

If $x_1, x_2 \in \mathbb{R}$ and $x_1 \neq x_2$, then the $(u_0(x_1), u_1(x_1))$ point is a self-crossing point. If $x_1 = x_2 \in \mathbb{R}$, then the $(u_0(x_1), u_1(x_1))$ point is a fourfold root. If x_1 is nonreal, then $x_1 = \bar{x}_2$ and the $(u_0(x_1), u_1(x_1))$ point is the supplementary point. From (21) and (22), we can obtain the following equations:

$$u_0 = x_1^2 x_2^2,$$
$$u_1 = -2x_1x_2(x_1 + x_2),$$
$$u_2 = (x_1 + x_2)^2 + 2x_1x_2,$$
$$1 = -2(x_1 + x_2)$$

for the parameters of the TDR-points. These equations yield the parameters of the TDR-points: $x_{1,2} = (-1 \pm d)/4$ (d is defined above) and the coordinates of the TDR-points:

$$u_0^* = \left(\frac{4u_2 - 1}{8}\right)^2, \quad u_1^* = \frac{4u_2 - 1}{8}.$$

Thus, if $3 > 8u_2$, then (u_0^*, u_1^*) is a self-crossing point:
if $3 = 8u_2$, then at (u_0^*, u_1^*) there is a fourfold root; and
if $3 < 8u_2$, then (u_0^*, u_1^*) is a complex supplementary point.

In these three cases, the D-curve is shown in fig. 5.

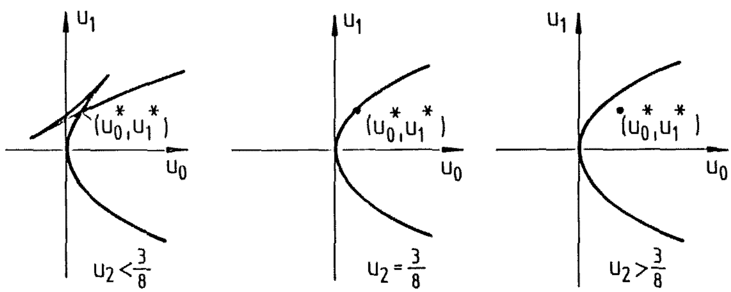

Fig. 5. "Ear–eye transition": the triangular region ("ear") – the vertices of which are the two cusps and the self-crossing point – collapses to a point (fourfold root) when $3 = 8u_2$, and then it continues itself as a complex supplementary point ("eye") inside the parabolic-like region.

From eqs. (23), we can see that the points of the set $\{(u_0, u_1) \in \mathbb{R}^2 : u_0 = u_1^2\}$ are the TDR-points on the parameter plane. We can consider this set as a curve parameterized by u_2.

$$u_0 = \left(\frac{u_2}{2} - \frac{1}{8}\right)^2, \quad u_1 = \frac{u_2}{2} - \frac{1}{8}$$

is the parametric form of the curve of the TDR-points. Thus, the curve of the TDR-points (TDR-curve) is a parabola on the parameter plane, with the equation $u_0 = u_1^2$.

If $u_2 < 3/8$, then the points of the curve belonging to this parameter are self-crossing points, if $u_2 = 3/8$, then the point of the curve belonging to this parameter is a fourfold root, and if $u_2 > 3/8$, then the points of the curve belonging to this parameter are the complex supplementary points of the D-curves.

Now we shall investigate the curve of the cusp points (C-curve). In a cusp point there is a triple root, c denotes this root, and the fourth root is $-1-3c$. Thus, in a cusp point

$$f(x) = (x-c)^3(x+1+3c). \tag{24}$$

From this equation, we can obtain the parametric form of the C-curve with the parameter c:

$$u_0 = -c^3(3c+1), \qquad u_1 = c^2(8c+3).$$

With the parameter u_2, the expression is more complicated. We can draw the C-curve: it has two cusp points at the parameters $c = 0$ and $c = -1/4$; thse points are the intersections with the TDR-curve. Figure 6 represents the C-curve and the TDR-

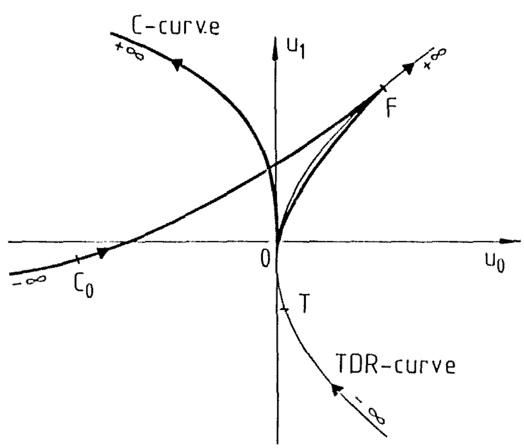

Fig. 6.

curve. The C-curve is parameterized by c, the TDR-curve is parameterized by u_2. From (24), $u_2 = -3c(2c+1)$; this is the connection between the parameters u_2 and c.

Using the notations in fig. 6, we briefly describe the motion of the TDR-point and the corresponding cusps.

(i) $u_2: -\infty \rightarrow 0$. The TDR point moves from infinity to T_0. To each value u_2, there belong two values of c, and the corresponding cusps move from infinity ($c = -\infty$) to C_0 ($c = -1/2$), and from infinity ($c = +\infty$) to the origin O ($c = 0$).

(ii) $u_2: 0 \rightarrow 3/8$. The TDR-point moves from T_0 to F. The corresponding cusps move from C_0 to F ($c = -1/4$) and from O to F, respectively.

(iii) $u_2: -3/8 \rightarrow +\infty$. The TDR-point moves from F to infinity. Cusp points do not exist in this case.

Finally, we shall study the H*-curve ($R = 0$). From (14), the parametric form of this curve is:

$$u_0 = -u_2 I^2 - I^4, \qquad u_1 = -I^2.$$

The parameter of the curve is I^2. From these formulas, we obtain the equation of the curve:

$$u_0 = u_1(u_2 - u_1).$$

Figure 7 represents this parabola. In the points of the parabola, $f(I) = f(-I) = 0$. That part of the parabola for which $I^2 < 0$ is the H-curve.

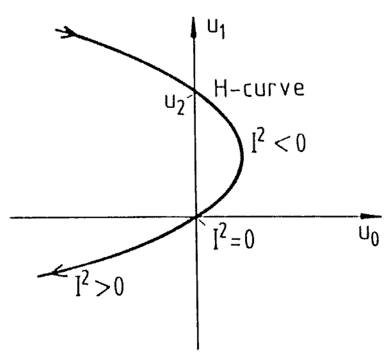

Fig. 7. The H*-curve.

Remark

The case when I is a complex number with a nonzero real part and a nonzero imaginary part (I^2 is nonreal) occurs only in the case $a_3 = 0$. If $a_3 = 0$, then the H*-curve is the $a_1 = 0$ straight line, which consists of two parts. The $\{(a_0, 0): a_0 < a_2^2/4\}$ half-line is the real part of the H*-curve (I^2 is real), and the $\{(a_0, 0): a_0 > a_2^2/4\}$ half-line is the nonreal part of the H*-curve (I^2 is nonreal). We can see the H*-curve in the case $a_3 = 0$ in fig. 8.

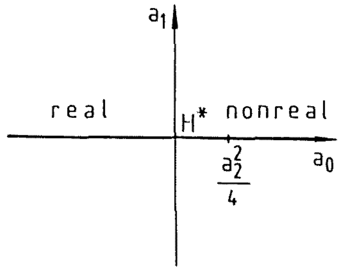

Fig. 8.

Appendix

In this appendix, we recall some algebraic terms and results [4, 14] concerning the roots of polynomials as well as the proof of the theorem.

Let f and g be polynomials:

$$f(x) = \sum_{i=0}^{n} a_i x^i, \qquad g(x) = \sum_{i=0}^{m} b_i x^i. \tag{A.1}$$

The *resultant* of the polynomials f and g is the determinant

$$R(f,g) = \begin{vmatrix} a_n & . & . & . & . & . & . & . & a_0 & 0 & . & . & . & 0 \\ 0 & a_n & . & . & . & . & . & . & . & a_0 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & 0 & a_n & . & . & . & . & . & . & . & a_0 \\ b_m & . & . & . & . & . & b_0 & 0 & . & . & . & . & . & 0 \\ 0 & b_m & . & . & . & . & . & b_0 & 0 & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & 0 & b_m & . & . & . & . & . & b_0 \end{vmatrix}. \tag{A.2}$$

The necessary and sufficient condition for the existence of a common root (that is, for the two polynomials f and g not to be relatively prime) is:

$$R(f, g) = 0.$$

The *discriminant* of the polynomial f is defined as:

$$D(f) = (-1)^{n(n-1)/2} \frac{1}{a_n} R(f, f'), \tag{A.3}$$

where f' denotes the derivative. The polynomial f has a multiple root if and only if the discriminant is zero.

THE ROUTH–HURWITZ CRITERION

The polynomial f is stable (that is, the real part of all its roots is negative) if and only if $\Delta_i > 0$ for $i = 1, 2, \dots, n$. Here, Δ_i are the ith principal minors of the Routh–Hurwitz matrix

$$\begin{pmatrix} a_{n-1} & a_{n-3} & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ a_n & a_{n-2} & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & a_2 & a_0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & a_3 & a_1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & a_4 & a_2 & a_0 \end{pmatrix}. \tag{A.4}$$

Here it is assumed that $a_n > 0$. In other words, the polynomial f is stable if and only if the following Routh–Hurwitz conditions are fulfilled:

$$\Delta_1 = a_{n-1} > 0, \tag{RH1}$$

$$\Delta_2 = a_{n-1}a_{n-2} - a_n a_{n-3} > 0, \tag{RH2}$$

$$\Delta_{n-1} > 0, \tag{RH$n-1$}$$

$$\Delta_n = a_0 \Delta_{n-1} > 0. \tag{RHn}$$

Now let us assume that we are in the stable region of the coefficient space $(a_0, a_1, \dots, a_{n-1})$, that is, all the conditions (RH1–RHn) are fulfilled. Varying the coefficients, there are two typical situations for the loss of stability. One is when a single real root changes its sign (from $-$ to $+$); this occurs when $a_0 = 0$. The other typical situation occurs when the real part of a conjugate pair of complex roots changes its sign (from $-$ to $+$). In the first case $\Delta_n = 0$, and we shall show that in the second case $\Delta_{n-1} = 0$, and this is the case when Hopf bifurcation takes place.

Now we formulate the theorem in another, equivalent form.

THEOREM

Let us assume that $a_0 \neq 0$. Then, $\Delta_{n-1} = 0$ if and only if there exists $I \in C$ for which I and $-I$ are roots of the polynomial f.

Proof

Let I and $-I$ be roots of f, that is,

$$f(I) = 0, \qquad f(-I) = 0. \tag{A.5}$$

Summing and subtracting these equations, we obtain an equivalent system of equations:

$$f(I) + f(-I) = 0, \qquad f(I) - f(-I) = 0. \tag{A.6}$$

These equations can be expressed in the forms:

$$a_0 + a_2 I^2 + \ldots = 0, \quad a_1 + a_3 I^2 + \ldots = 0. \tag{A.7}$$

This means that the even and odd parts are separately zero. Recalling that two algebraic equations can have a common solution if and only if the resultant is zero, we obtain the following necessary and sufficient condition

$$R(f^-, f^+) = \begin{vmatrix} a_1 & a_3 & a_5 & . & . & 0 \\ 0 & a_1 & a_3 & a_5 & . & 0 \\ \ldots & & & & & \ldots \\ a_0 & a_2 & a_4 & . & . & 0 \\ 0 & a_2 & a_4 & . & & 0 \end{vmatrix} = 0. \tag{A.8}$$

Rearranging the rows of the determinant, we obtain

$$R(f^-, f^+) = \begin{vmatrix} a_1 & a_3 & a_5 & & 0 \\ a_0 & a_2 & a_4 & & 0 \\ 0 & a_1 & a_3 & a_5 & 0 \\ 0 & a_0 & a_2 & a_4 & 0 \\ \ldots & & & & \ldots \\ 0 & & & & a_n \end{vmatrix} = 0.$$

It is obvious that this is just the $(n-1)$th Routh–Hurwitz condition:

$$\Delta_{n-1} = 0. \tag{RH$n-1$}$$

This proves the theorem. □

Now we mention here another criterion of stability of polynomials with positive coefficients. (Obviously, if the polynomial is stable and $a_n > 0$, then $a_i > 0$ for all i.)

THE LIENNARD–CHIPARD CRITERION [15]

The polynomial f is stable if and only if $a_i > 0$ for $i = 0, 1, \ldots, n$, and $\Delta_{n-1} > 0$, $\Delta_{n-3} > 0$, $\Delta_{n-5} > 0, \ldots$.

We apply this criterion for the cases $n = 3, 4, 5$ and restrict ourselves to positive coefficients ($a_i > 0$, $i = 0, 1, \ldots, n$).

For cubic polynomials, the only condition of stability is $\Delta_2 > 0$. In the case of the quartic equation, another condition is also relevant: $\Delta_3 > 0$. In the case of the quintic equation, the relevant conditions are $\Delta_2 > 0$ and $\Delta_4 > 0$.

These results help us to reveal the location of the stability region of the polynomial f. This region is in the positive orthant of the space of the coefficients $a_0, a_1, \ldots, a_n$. In the case of the quartic equation, the stability region is bordered by the surface $\Delta_3 = 0$. In the case of the quintic equation, the set defined by the condition $a_i > 0$ ($i = 0, \ldots, 5$) and $\Delta_4 > 0$ is not connected; one of its components (in which $\Delta_2 > 0$) is identical with the stable domain.

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